2-microlocal Analysis and Applications

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Projet Fractales

INRIA

Part 1: Deterministic 2-microlocal analysis

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Part 1: Deterministic 2-microlocal analysis

Recalls on regularity exponents

Measuring the *local* regularity of functions is important in many areas of mathematics and applications.

Local regularity characterization is most often based on the pointwise Hölder exponent (e.g. in multifractal analysis).

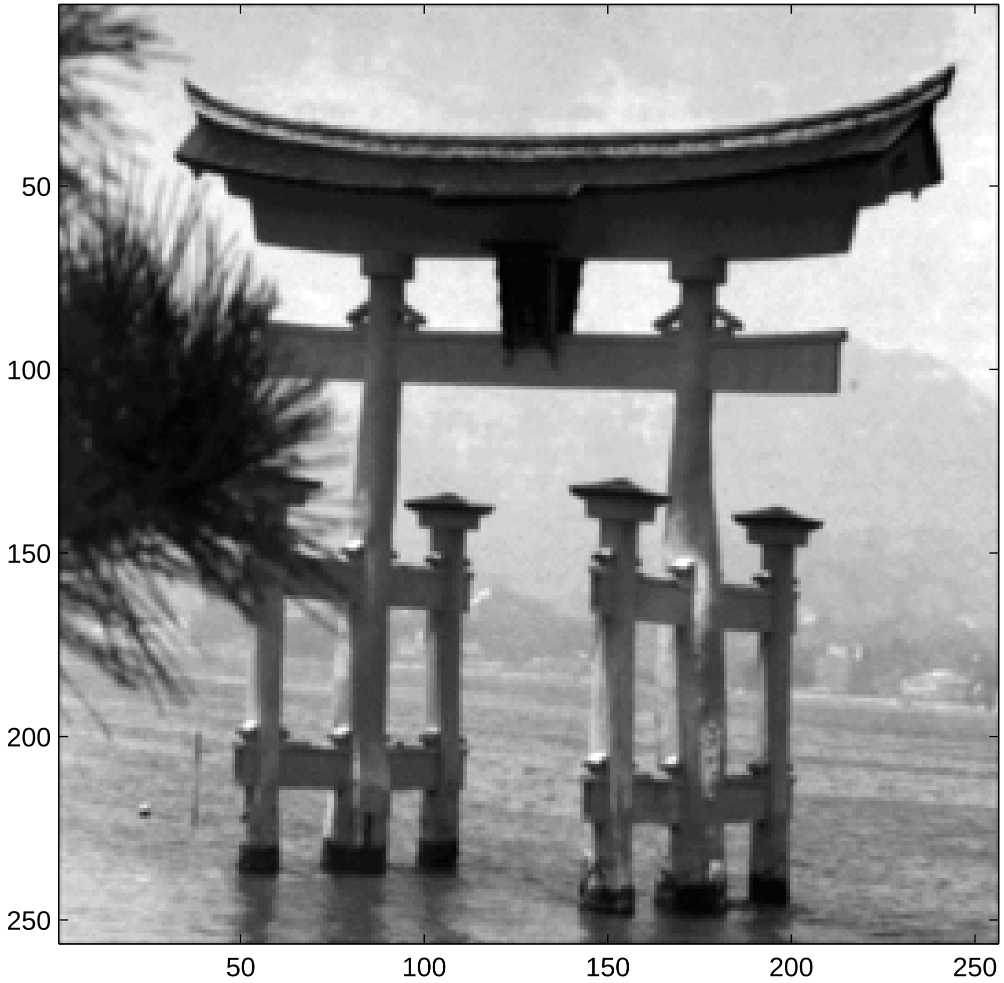
The pointwise Hölder exponent in applications

The pointwise Hölder exponent has been used to:

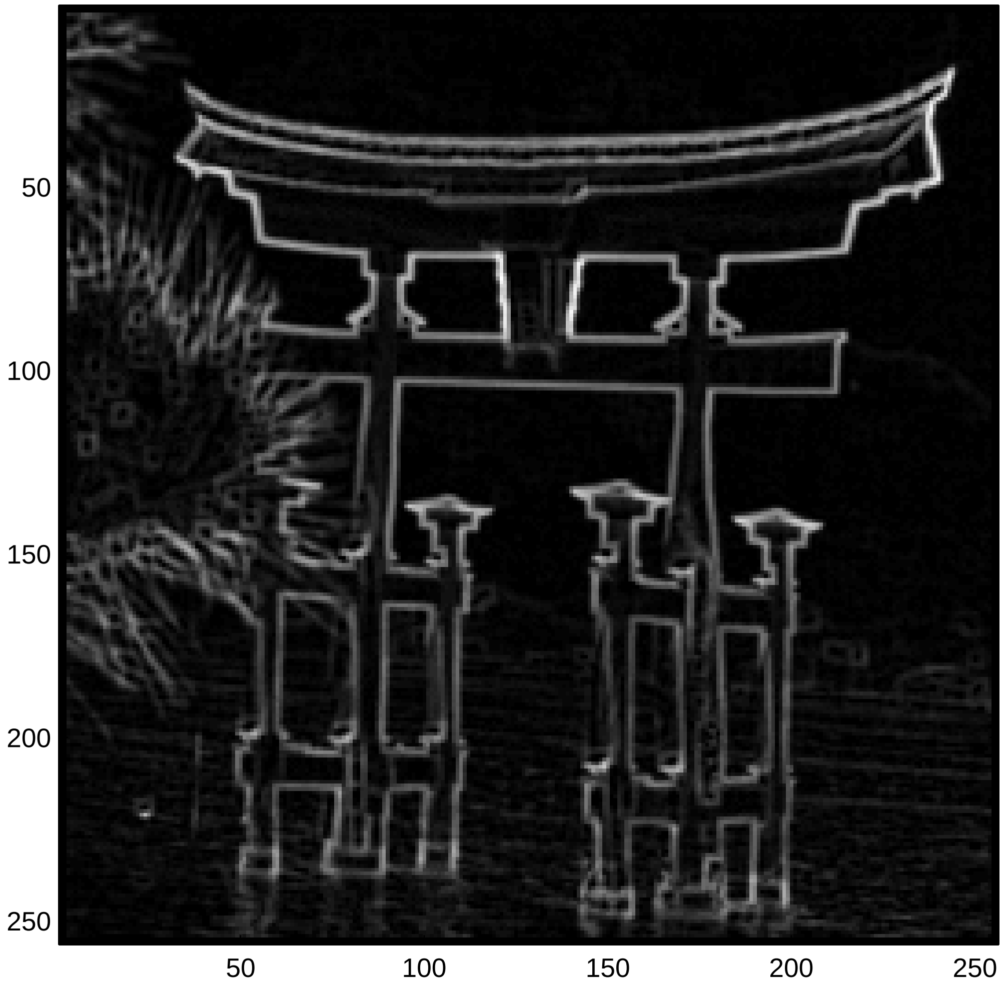
1. detect contours in images,
2. perform data interpolation,
3. denoise signals and images.

In addition, the distribution of these exponents (multifractal analysis) has been used to predict the queuing behaviour of Internet traffic, to characterize ECG signals, and more.

Example 1: Pointwise Hölder exponent-based edge detection



Original "door" image.



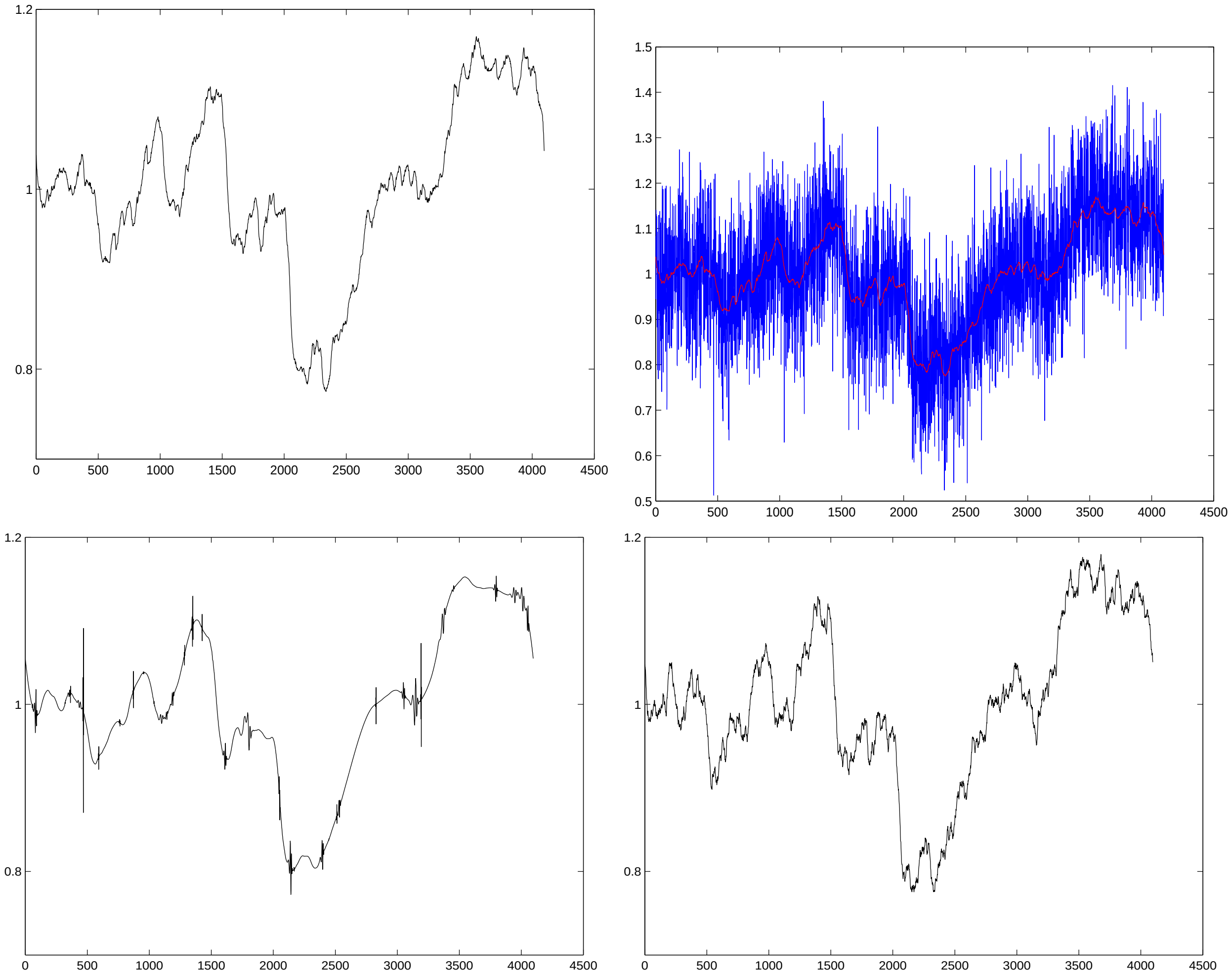
Edges detected on the "door" image using the pointwise Hölder exponent (image obtained with *FracLab*).

Example 2: Image interpolation

Zoom by a factor 3 on a detail of the ”door” image: **Bicubic interpolation**. Remark the many artifacts, specially around edges.

Zoom by a factor 3 on a detail of the "door" image: **Pointwise Hölder exponent-based interpolation** (image obtained with *FracLab*).

Example 3: Pointwise Hölder Exponent-based signal denoising



Top: Original signal (left), superimposed with noisy version (right). Bottom: Denoising using the hard-thresholding wavelet (left) and Hölder exponent (right) schemes. Thresholding oversmooths and introduces oscillations (results obtained with *FracLab*).

Definition of the pointwise Hölder exponent

Definition: The pointwise Hölder exponent at x_0 of a continuous function f: $\mathbb{R} \to \mathbb{R}$ is the number $\alpha = \alpha^f(x_0)$ such that :

- $\forall \gamma < \alpha, \lim_{h \to 0} \frac{|f(x_0+h)-P(h)|}{|h|^\gamma} = 0,$

- if $\alpha < +\infty, \forall \gamma > \alpha, \limsup_{h \to 0} \frac{|f(x_0+h)-P(h)|}{|h|^\gamma} = +\infty$

where P is a polynomial of degree not larger than the integer part of α.

(This definition has to be adapted if α is an integer.)

Particular case of continuous non-differentiable functions

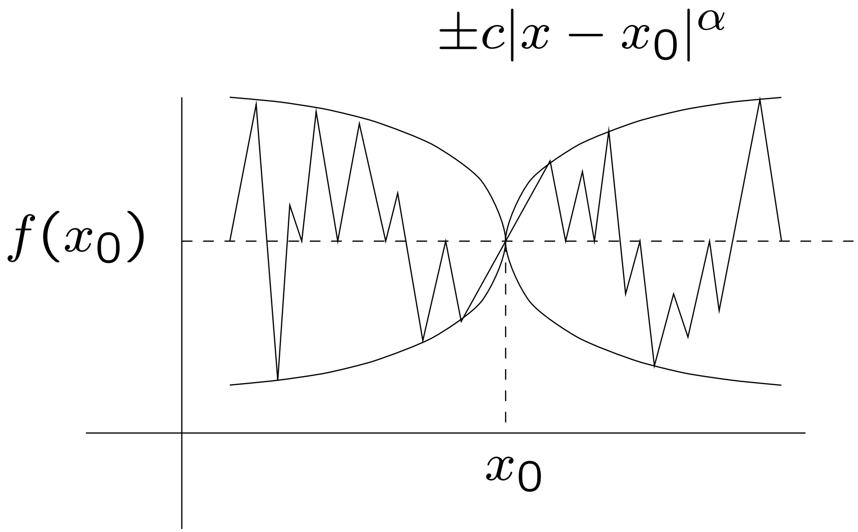
When $0 < \alpha < 1$, equivalent definitions read:

$$\alpha = \sup\{\beta : \exists c, \rho_0 > 0, \forall \rho < \rho_0 \sup_{x,y \in B(x_0,\rho)} |f(x) - f(y)| \leq c\ \rho^\beta\}$$

or

$$\alpha = \liminf_{h \to 0} \frac{\log |f(x_0 + h) - f(x_0)|}{\log |h|}$$

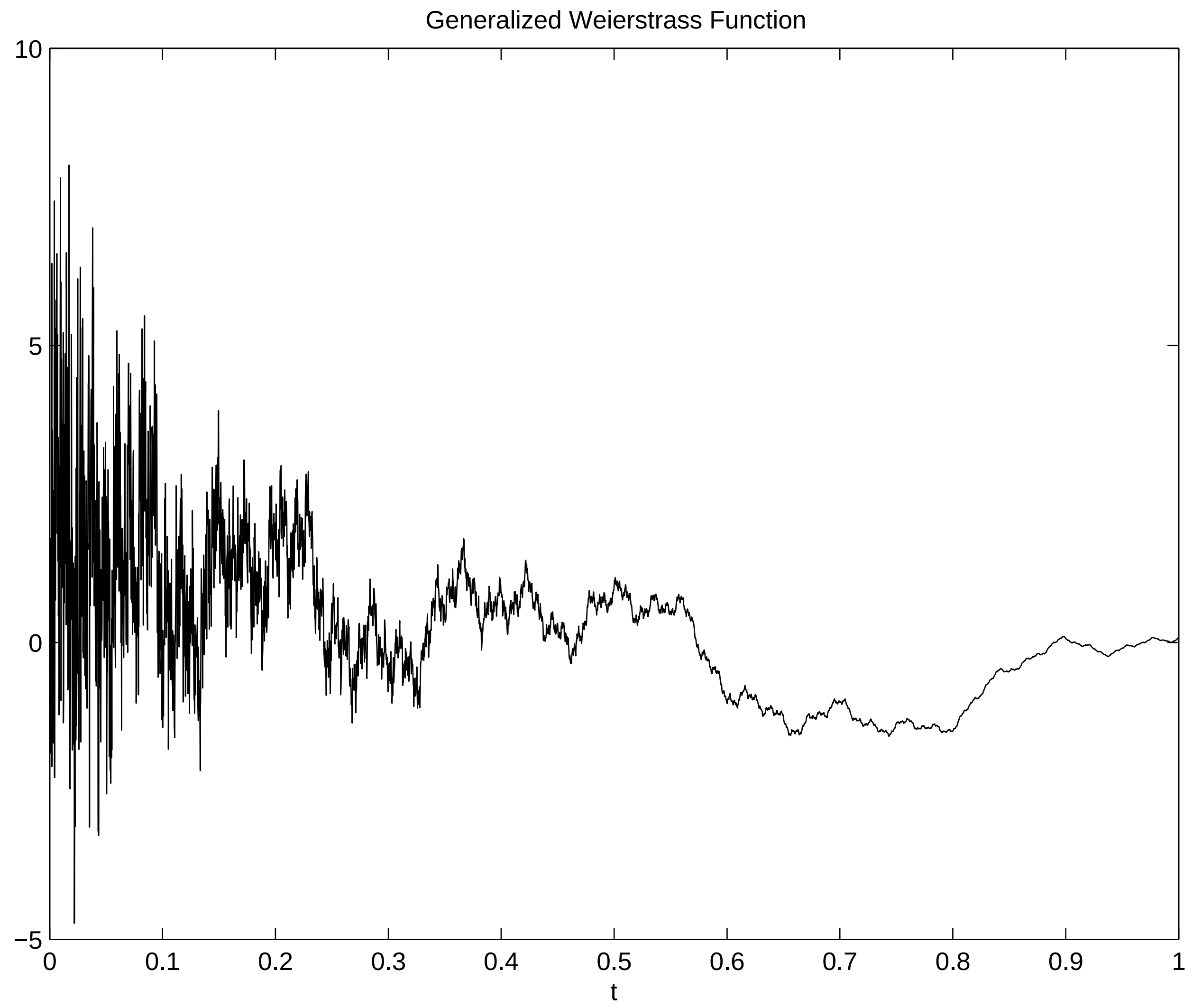
The intuitive notion of pointwise Hölder Exponent



α has certain nice properties:

- (NP1) It captures to some extent the intuitive perception of regularity.

- (NP2) It is a versatile tool: The *pointwise Hölder function* of a continuous function f (*i.e.* the function $x \mapsto \alpha^f(x)$) is able to record fine changes in the regularity of f.



NP1: A signal with $\alpha(t) = \alpha_l(t) = t$ for all t in $[0, 1]$ (signal obtained with *FracLab*).

NP2: The structure of the pointwise Hölder function

Theorem (Daoudi/Lévy Véhel/Meyer; Jaffard):

Let s be a function from $[0,1]$ to $[0,1]$. The following conditions are equivalent:

- s is the pointwise Hölder function of a continuous function.
- There exists a sequence of continuous functions $(s_n)_{n\geq 1}$ such that:

$$s(x) = \liminf_{n\to\infty} s_n(x)$$

Example: $s = 1\!\!1_{\mathbb{Q}}(x)$ *is* the pointwise Hölder function of a continuous function, while $1 - s$ is *not*.

α_p has certain undesirable properties:

- $\alpha_p(x)$ does not contain all the regularity information.

  Ex. :

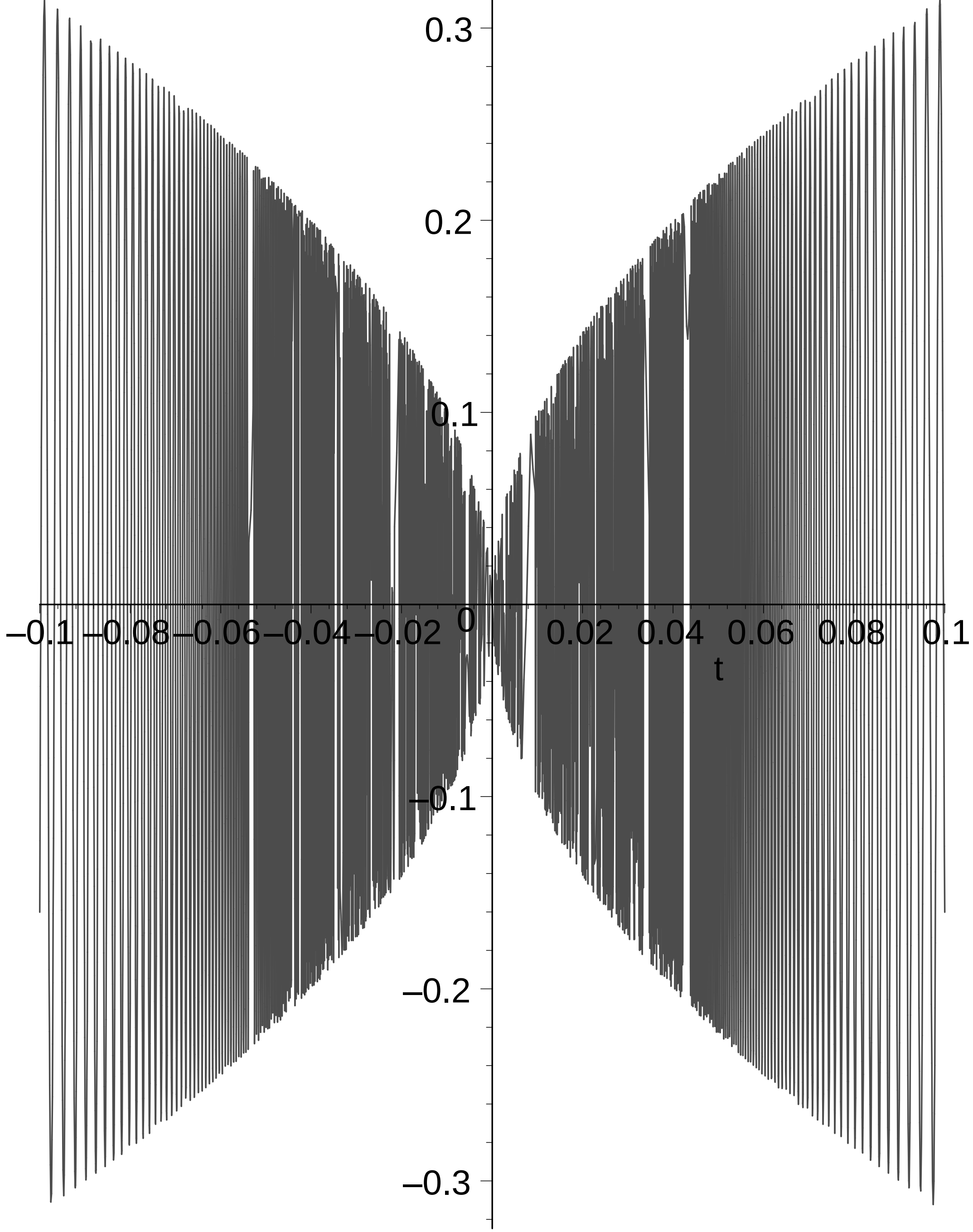
  $$\text{"cusp function"}: \ |x|^\gamma \qquad \gamma > 0 \qquad \alpha_p(0) = \gamma$$

  $$\text{"chirp function"}: \ |x|^\gamma \sin \frac{1}{|x|^\beta} \quad \gamma > 0, \beta > 0 \quad \alpha_p(0) = \gamma$$

- α_p is not stable through the action of (pseudo-)differential operators. For instance, one can only ensure in general that:

  $$\alpha_p^{X'} \leq \alpha_p^X - 1$$

  (chirp : $\alpha_p^{X'}(0) = \alpha_p^X(0) - 1 - \beta$)



Chirp $|x|^{\gamma} \sin \frac{1}{|x|^{\beta}}$

This means for instance that:

- The multifractal spectrum does not give a complete description of the regularity structure of a function/measure.

- The multifractal spectrum of X' cannot be deduced from the one of X.

$\longrightarrow$ We need to supplement α_p with additional regularity exponents.

The local Hölder exponent

Definition (Guiheneuf/Lévy Véhel):

Let

$$\alpha_l(x,\varepsilon) = \sup\{\alpha : f \in C^\alpha(B(x,\varepsilon))\}$$

where $B(x,\varepsilon)$ is the ball centered at x with radius ε. Then the local Hölder exponent of f at x is

$$\alpha_l(x) = \lim_{\varepsilon\to 0} \alpha_l(x,\varepsilon)$$

When $0 < \alpha < 1$, this is equivalent to:

$$\alpha_l(x_0) = \sup\{\beta : \exists c, \rho_0 > 0, \forall \rho < \rho_0 \sup_{x,y\in B(x_0,\rho)} |f(x) - f(y)| \le c\,|x-y|^\beta\}$$

The local Hölder Exponent: Pros and Cons

Advantages of the local Hölder exponent:

- α_l is sensitive to oscillations:

  $\alpha_l(0) = \frac{\gamma}{1+\beta}$ for $f(x) = |x|^\gamma \sin \frac{1}{|x|^\beta}$

- α_l is stable through integro-differentiation

Drawback of the local Hölder exponent:

The evolution of the local Hölder function $x \mapsto \alpha_l(x)$ is rather constrained: In particular that the local Hölder exponent will be constant for many "interesting" multifractal functions/measures.

The structure of the local Hölder function

Theorem (Guiheneuf/Lévy Véhel):

Let s_l be a function from $[0,1]$ to $[0,1]$. The following conditions are equivalent:

- s_l is the local Hölder function of a continuous function.
- s_l is lower semi-continuous.

Thus, one cannot detect a "regular" point in an "irregular" background with the help of α_l

Relation between the pointwise and local Hölder functions

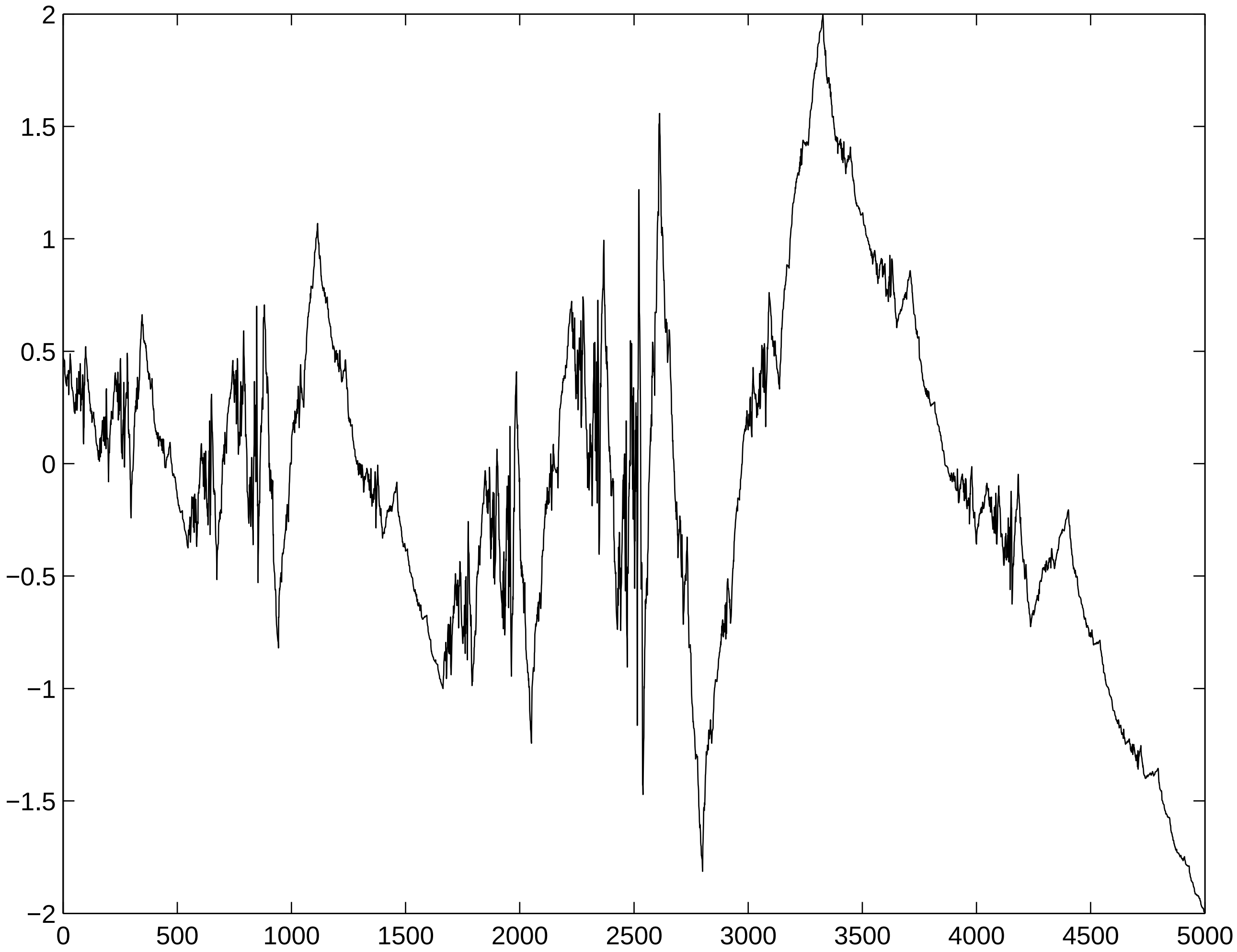
Theorem (Lévy Véhel/Seuret):

Let s be a function from $[0,1]$ to $[0,1]$. Assume s is the liminf of a sequence of continuous functions. Let s_l be a lower semi-continuous function. Assume that $s_l \leq s$. Then there exists a continuous function f such that:

- $\alpha_l(x) = s_l(x) \ \forall x$
- $\alpha(x) = s(x) \ \forall x \notin E$, where E is an everywhere dense set with Hausdorff dimension equal to 0.

Theorem (Lévy Véhel/Seuret):

Let f be a continuous function. Then its local and pointwise Hölder functions coincide at least on an everywhere dense set with Hausdorff dimension equal to 0.



A signal with $\alpha(t) \neq \alpha_l(t)$ for all t outside a set of zero Hausdorff dimension (signal obtained with *FracLab*).

Other regularity exponents

The couple (α_p, α_l) does not encompass all the local regularity information. Other exponents have been defined, which include:

- the oscillating exponent,
- the weak scaling exponent,
- and the chirp exponent.

The oscillating exponent

Definition (Jaffard): Let $f \in L^1_{loc}(\mathbb{R})$. Denote by $\alpha_t(x_0)$ the pointwise Hölder exponent of a fractional integral of order t of f at x_0. Then the *oscillating exponent* of f at x_0, denoted β_o, is defined as:

$$\beta_o(x_0) = \left(\frac{\partial}{\partial t}\alpha_t(x_0)\right)_{t=0^+} - 1.$$

Thus $\beta_o + 1$ measures the rate of increase of α through infinitesimal integration.

It is easy to show that β_o is always non-negative.

"Nice" functions, such as the cusp $|x|^\gamma$ have $\beta_o = 0$.

Functions that "oscillate", such as the chirp $|x|^\gamma \sin \frac{1}{|x|^\beta}$, have $\beta_o > 0$.

A function f is said to be oscillating at x_0 if $\beta_o^f(x_0) > 0$.

The weak scaling exponent

Definition (Meyer)

Let f be a function in $L^1_{loc}(\mathbb{R})$, and denote by $f^{(-l)}$ a primitive of f of order l. Then

$$\beta_w(x_0) = \sup\{s : \exists n,\ f^{(-n)} \in C^{s+n}_{x_0}\}.$$

This exponent thus characterizes the asymptotic behaviour of the function after a large number of integrations.

The chirp exponent

Definition (Meyer)

Let f be a function in $L^1_{loc}(\mathbb{R})$, and denote by $f^{(-l)}$ a primitive of f of order l. f is called a (h, β_c)-type chirp at x_0 if

$$\forall n \in \mathbb{N},\ f^{(-n)} \in C^{h+n(1+\beta_c)}_{x_0}.$$

This exponent measure, in a very precise sense, the oscillatory content of the function.

Examples

- $f(x) = |x|^\gamma$: $\beta_w(0) = \alpha_l(0) = \alpha(0)$ and $\beta_c(0) = \beta_o(0) = 0$.

- Weierstrass function: $\beta_w(x) = \alpha_l(x) = \alpha(x)$ and $\beta_c(x) = \beta_o(x) = 0$ for all x.

- $f(x) = |x|^\gamma \sin \frac{1}{|x|^\beta}$, $\gamma > 0$, $\beta \geq 0$: $\alpha(0) = \gamma$, $\beta_w(0) = \infty$, $\beta_c(0) = \beta_o(0) = \beta$ and $\alpha_l(0) = \frac{\gamma}{1+\beta}$.

- Fractal Interpolation Functions/Multiplicative Cascades:

  Pointwise exponent: α is everywhere discontinuous and ranges in an interval $[\alpha_m, \alpha_M]$. Furthermore, all the level sets of the function $x \mapsto \alpha(x)$ are everywhere dense.

  β **exponents:** $\beta_w = \alpha$, $\beta_c = \beta_o = 0$ at all points.

  Local exponent: $\alpha_l = \alpha_m$ at all points.

Note that only α_l indicates that "something is going on", i.e. that α does not give exhaustive information: Fractal interpolation functions and cascades are not oscillating functions, but they are "multisingular" (see below for a definition) at each point.

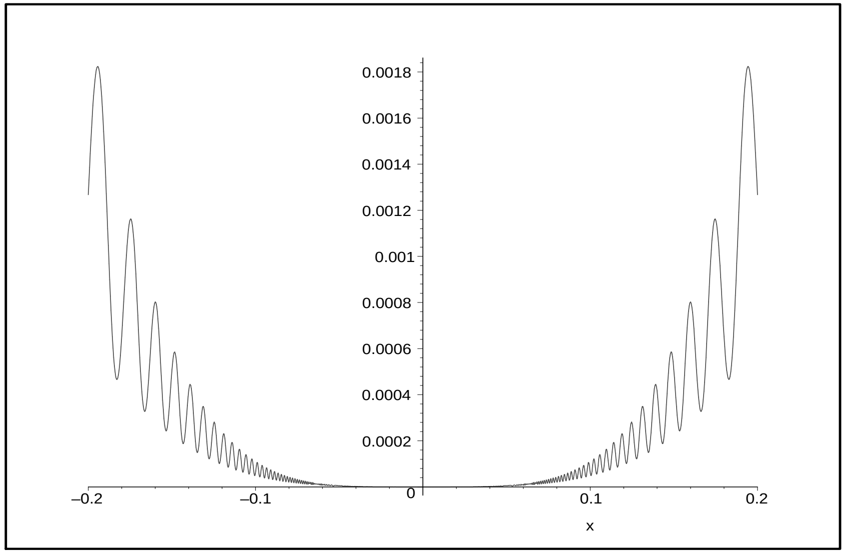
Towards 2ml Analysis (I)

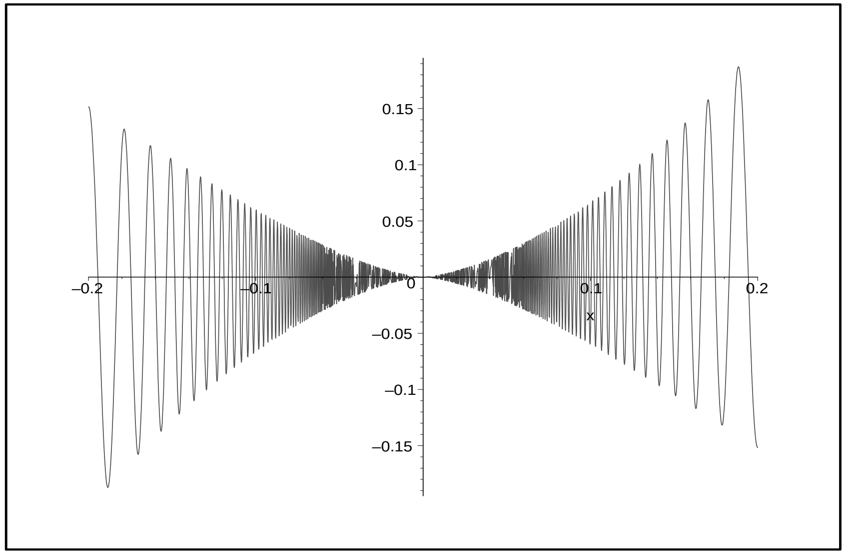
In some applications, no combination of these exponents gives sufficient information.

Example: Consider the exponents (α, β_o). This couple measures the current pointwise regularity, and how it will evolve under infinitesimal integration. What about finite order integration or differentiation ?

For instance, a function could be oscillation-free, but its derivative could be oscillating.

Example: $f(x) = x^{4.1} + x^{4.5} \sin(1/x^2)$.





An oscillation-free function whose derivative oscillates.

Towards 2ml Analysis (II)

The main point is that the set of exponents defined above do not allow to predict the pointwise Hölder exponent of (fractional) integrals or derivatives of f at a given point (except in simple cases such as the cusp/chirp).

2-microlocal analysis is a clever way to gather the knowledge which is necessary to compute such exponents.

Recalls: Littlewood-Paley analysis

As introduced by J.M. Bony, the concept of 2-ml spaces relies on a Littlewood-Paley analysis:

$$\text{Let } \varphi \in \mathcal{S}(\mathbb{R}),\ \hat{\varphi} \ = \ 1 \qquad \text{on} \qquad \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$$= \ 0 \quad \text{outside} \quad [-, 1]$$

Let $f \in \mathcal{S}'(\mathbb{R})$. The *Littlewood-Paley* analysis of f is:

$$\begin{cases} S_0 f & = \ \varphi * f \\ \Delta_j f & = \ \Psi_j * f \qquad j \in \mathbb{N} \end{cases}$$

where

$$\Psi_j(x) \ = \ 2^{j+1}\varphi\left(2^{j+1}x\right) - 2^j\varphi\left(2^j x\right)$$

One has the fundamental decomposition:

$$f \ = \ S_0 f + \sum_{j=0}^{\infty} \Delta_j f$$

2ml Analysis: Original definition

Definition (Bony):

Let $x_o \in \mathbb{R}, (s, s') \in \mathbb{R}^2$. $f \in \mathcal{S}'(\mathbb{R})$ belongs to the 2ml space $C^{s's}_{x_0}$ if:

$$\begin{array}{rrcl} \exists\ C : & |S_0 f(x)| & \leq & C[1 + |x - x_0|]^{-s'} \\ \forall\ j \in \mathbb{N} & |\Delta_j f(x)| & \leq & C 2^{-js}[1 + 2^j |x - x_0|]^{-s'} \end{array}$$

Fundamental property of 2-ml spaces

Theorem (Bony)

$$f \in C^{s,s'}_{x_0} \Longleftrightarrow f^{(n)} \in C^{s-n,s'}_{x_0}$$

In the sequel, we will be considering "local" versions of 2-ml spaces: By abuse of language, f belongs to the 2ml space $C^{s's}_{x_0}$ will mean that f coincides on a neighbourhood of x_0 with a distribution g that belongs to $C^{s's}_{x_0}$.

Some basic properties of 2-microlocal spaces

$\forall x_0 \in \mathbb{R}$,

- $t \leq s$ and $t + t' \leq s + s' \Rightarrow C_{x_0}^{s,s'} \subset C_{x_0}^{t,t'}$.

- $\forall s > 0$, $C_{x_0}^{s} \subset C_{x_0}^{s,-s}$.

- $\forall (s, s')$ with $s + s' > 0$, $C_{x_0}^{s,s'} \subset C_{x_0}^{s}$.

Recalls: Wavelet analysis. Discrete case

There exists a wavelet-based characterization of 2-ml spaces. Before presenting it, we recall the definition of the wavelet transform.

Let ψ be a function in the Schwartz class $\mathcal{S}(\mathbb{R})$ such that $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)\}_{(j,k)\in\mathbb{Z}^2}$ forms an orthonormal basis of $L^2(\mathbb{R})$.

The discrete wavelet coefficients of f are defined as:

$$d_{j,k} = 2^j \int f(x)\psi(2^j x - k)dx.$$

One has:

$$f(x) = \sum_{(j,k)\in\mathbb{Z}^2} d_{j,k}\psi(2^j x - k)$$

Recalls: Wavelet analysis. Continuous case

Let Ψ be an $L^2(\mathbb{R})$ function with zero mean, satisfying the "admissibility condition".

The continuous wavelet transform (or coefficients) of f is defined, for $a > 0$ and $b \in \mathbb{R}$, by

$$C(a,b) = \frac{1}{a} \int f(x) \Psi \left(\frac{x-b}{a} \right) dx.$$

2-ml spaces: Wavelet-domain definition

Theorem (Jaffard)

$$f \in C^{s,s'}_{x_0} \iff$$

$$\exists\ C : \forall\ j,k \quad : \quad \begin{array}{l} |x_0 - k2^{-j}| \leq 1, \\ |d_{j,k}| \leq C 2^{-js}(1 + |k - 2^j x_0|)^{-s'} \end{array}$$

$$\iff$$

$$\exists\ C : \forall\ a > 0, \quad \forall\ b, \quad \begin{array}{l} |b - x_0| \leq 1, \\ |C(a,b)| \leq C a^s \left(1 + \dfrac{|b - x_0|}{a}\right)^{-s'} \end{array}$$

2-ml spaces: Time-domain definition

Theorem (Kolwankar/Levy Vehel/Seuret)

Let $x_0 \in \mathbb{R}$, and s, s' be two real numbers satisfying $s+s' > 0$, $s+s' \notin \mathbb{N}$, and $s' < 0$ (and thus $s \geq 0$). Let $m = [s+s']$.

A function $f : \mathbb{R} \to \mathbb{R}$ belongs to $C^{s,s'}_{x_0}$ if and only if its m^{th} derivative exists around x_0, and if there exist $\delta > 0$, a polynomial P of degree smaller than $[s] - m$, and a constant C, that verify

$$\left| \frac{\partial^m f(x) - P(x)}{|x - x_0|^{[s]-m}} - \frac{\partial^m f(y) - P(y)}{|y - x_0|^{[s]-m}} \right|$$

$$\leq C|x-y|^{s+s'-m}(|x-y| + |x-x_0|)^{-s'-[s]+m}$$

for all x, y such that $0 < |x - x_0| < \delta$, $0 < |y - x_0| < \delta$.

Time-domain definition. Particular case of continuous non-differentiable functions

When $0 \leq s+s' < 1, s \geq 0, -1 \leq s' \leq 1$, the characterization takes the much simpler form:

$$|f(x) - f(y)| \leq C|x-y|^{s+s'}(|x-y| + |x-x_0|)^{-s'}$$

The 2-microlocal domain and the 2ml frontier

Stating that the Hölder exponent of f at x is α is the same as saying that α is the supremum of the real numbers s such that f belongs to the pointwise Hölder space C_x^s.

We seek a notion corresponding to α for $C^{s,s'}$ spaces. This is more complicated, since there are two exponents, and there is no total order in $\mathbb{R}^2$.

Definition
The *2ml domain* of f at x_0 is the set:

$$\{(s,s') \in \mathbb{R}^2 : f \in C^{s,s'}_{x_0}\}$$

Proposition
The 2ml domain is a convex subset of $\mathbb{R}^2$.

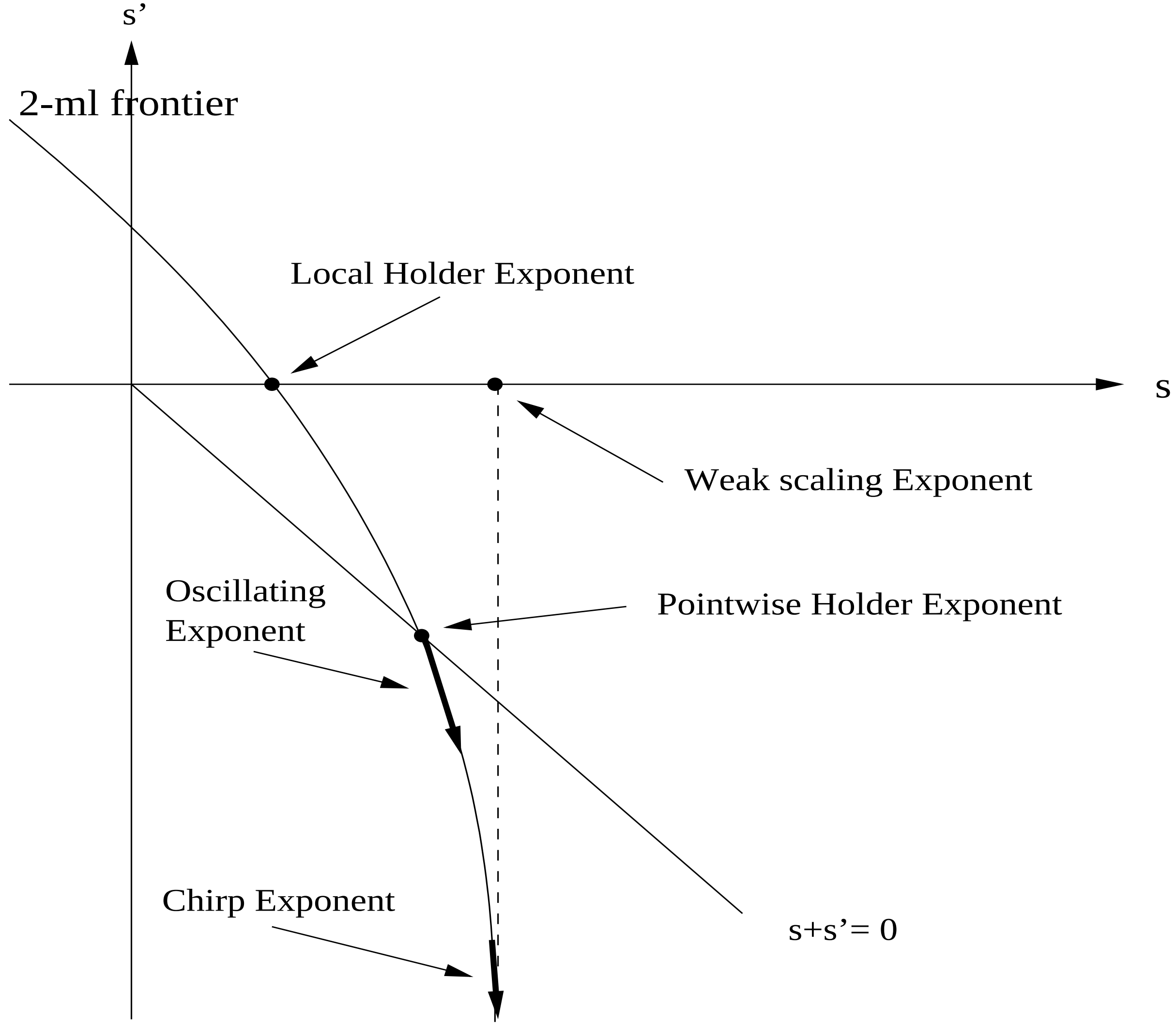
Definition
The *2ml frontier* of f at x_0 is the curve

$$\begin{aligned} &\mathbb{R} \to \mathbb{R} \\ &s' \mapsto s(s') = \sup\{u : f \in C^{u,s'}_{x_0}\} \end{aligned}$$

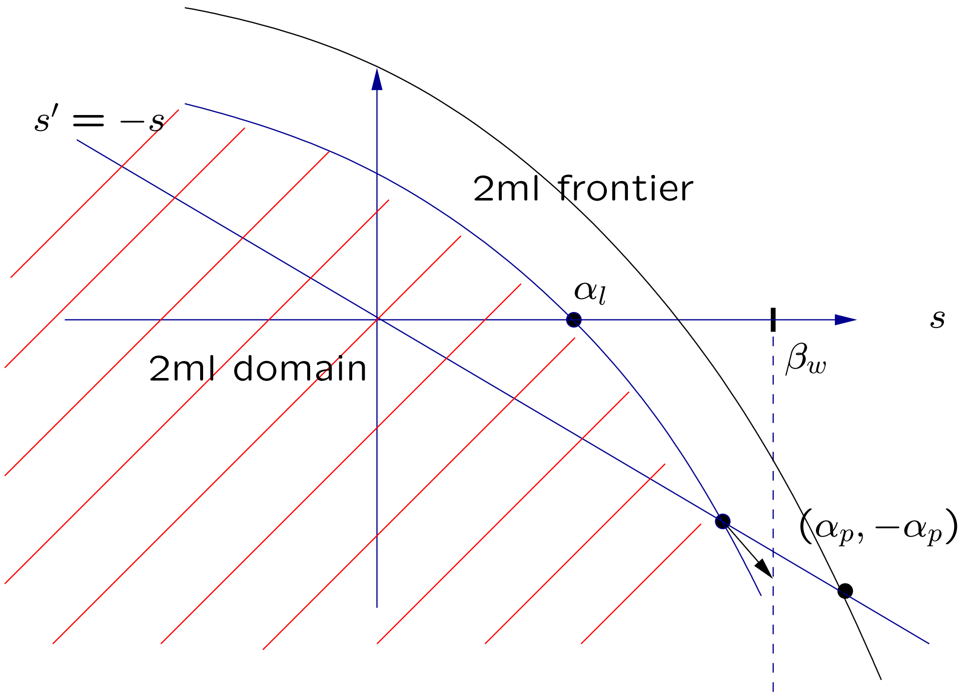
Proposition
The 2ml frontier is non increasing, concave, with slope everywhere between -1 and 0.

2-ml frontier and exponents in the (s, s')-plane

Theorem: Assume $\alpha_l(x) > 0$. Then, the regularity exponents defined above are as indicated on the figure below.



The 2ml frontier, the stability of 2ml spaces through integration and the tracking of α



Through integration, the blue frontier is translated along the $s-$axis. The new pointwise Hölder exponent is obtained by taking the intersection between the new frontier (in black) and the line $s = -s'$.

2-ml frontier parameterization

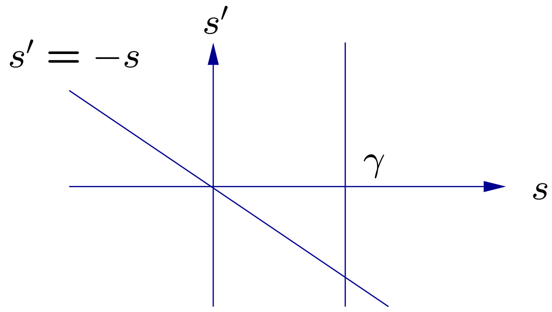
Since the frontier may be vertical in the (s, s') plane, one cannot use s as the free variable.

For reasons that will appear shortly, a convenient parameterization is to define the frontier as the curve:

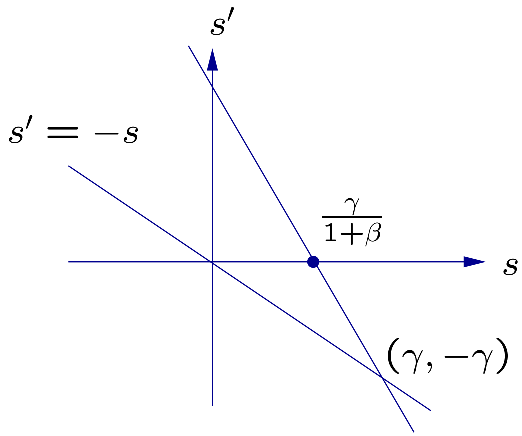
$$s' \to \sigma(s'), \text{ where } \sigma = s + s'$$

Examples of 2-ml frontiers

- $f(x) = |x|^\gamma, \gamma > 0, x_0 = 0.$

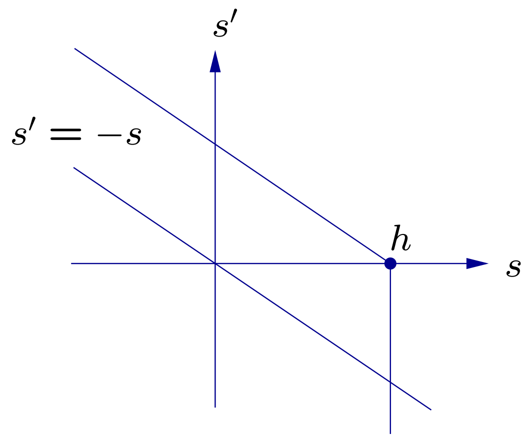


- $f(x) = |x|^\gamma \sin \frac{1}{|x|^\beta}, \gamma > 0, \beta > 0, x_0 = 0.$

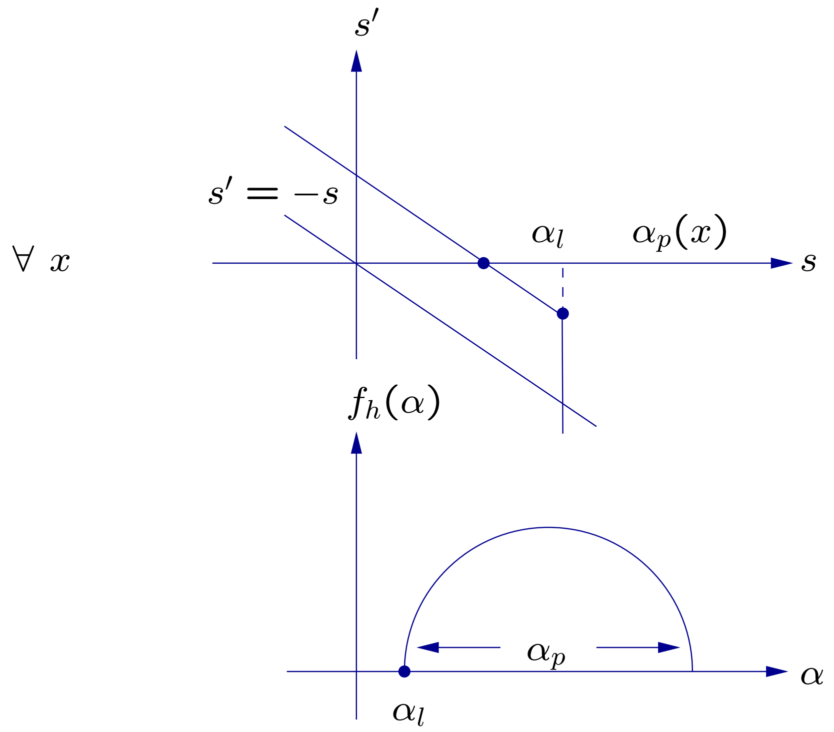


- Weierstrass function $W(x) = \sum_{n=1}^{\infty} \lambda^{-nh} \sin \lambda^n x$.

$\forall\ x$



- Fractal Interpolation Function/Multiplicative cascade (deterministic multinomial measure)



2-ml frontier (top), Multifractal spectrum (bottom)

The 2ml Formalism

The conditions to be verified for checking whether a distribution belongs to some 2-ml space are not simple.

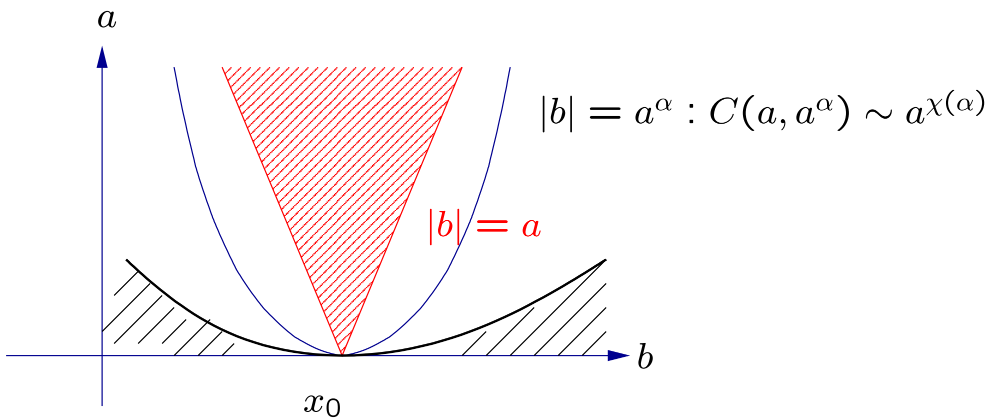
It occurs that the 2-ml frontier of any distribution may be obtained as the Legendre transform of function which is easier to compute, called the *2-ml spectrum*.

Two different 2-ml spectra may be defined, depending on whether one works in the wavelet or the time domain.

We concentrate here on the wavelet-based 2-ml spectrum.

The 2-ml spectrum: Intuitive idea

The basic observation is that both the *magnitude* and the *location* of the wavelet coefficients are important to decide whether a distribution belongs or not to some 2-ml space.



The three domains used to define the 2-ml spectrum: (I) "above" the point x_0 (red area), (II) all the curves $|b| = a^\alpha$ for $\alpha \in (0, 1)$, and (III) the region "below" all such curves (black zone). On a curve $|b| = a^\alpha$, one one define $\chi(\alpha)$ as the exponent such that the largest coefficients $C(a, a^\alpha)$ are "of the order" of $a^{\chi(\alpha)}$, with a similar approach for the two other regions.

The 2-ml spectrum: Definition

Let, for $0 < \alpha < 1$:

$$
\begin{aligned}
\chi^{\varepsilon}(\alpha) &= \liminf_{a \to 0} \inf_{\alpha-\varepsilon \le \beta \le \alpha+\varepsilon} \frac{\log |C(a, x_0 \pm a^{\beta})|}{\log a} \\
\chi(\alpha) &= \lim_{\varepsilon \to 0^+} \chi^{\varepsilon}(\alpha) \\
\chi(0) &= \lim_{\alpha \to 0^+} [\sup\{\gamma : \exists b_0 > 0, a^{\alpha} \le (b - x_0) < b_0 \\
&\Longrightarrow |C(a, x_0 + b)| \le K_{\alpha} a^{\gamma}\}] \\
\chi(1) &= \lim_{\alpha \to 1^-} [\sup\{\gamma : \exists b_0 > 0, (b - x_0) < \min(b_0, a^{\alpha}) \\
&\Longrightarrow |C(a, x_0 + b)| \le K_{\alpha} a^{\gamma}\}]
\end{aligned}
$$

$\chi : [0, 1] \to \mathbb{R}^+ \cup \{+\infty\}$ is called the *2-ml spectrum* of f at x_0.

Basic properties of χ

Proposition

Let $\tilde{\chi}$ denote the convex envelop of χ. Then:

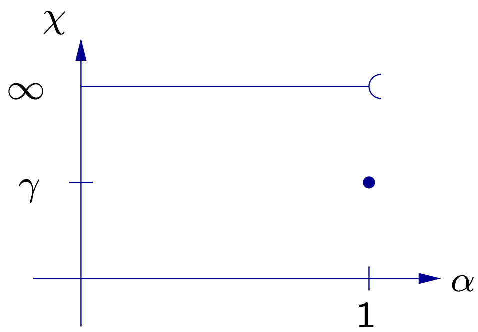
$\tilde{\chi}(0) = \chi(0)$ and $\tilde{\chi}(1) = \chi(1)$.

Proposition

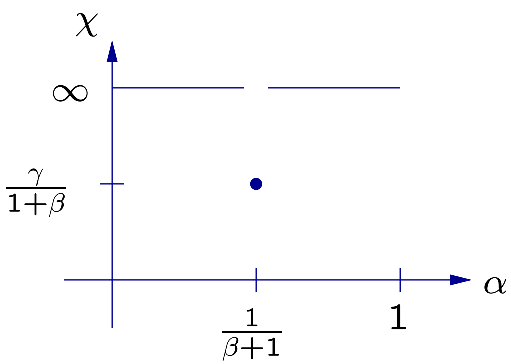
The function $\rho \mapsto \chi(\rho)$ is lower semi-continuous.

Examples of 2-ml spectra

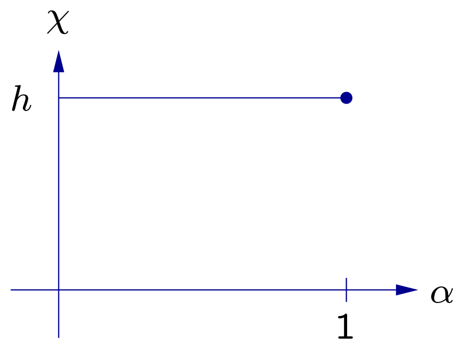
$f(x) = |x|^\gamma$:



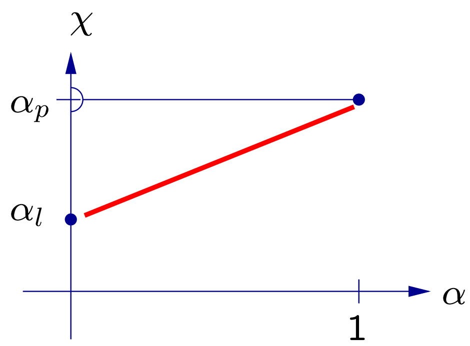
$f(x) = |x|^\gamma \sin \frac{1}{|x|^\beta}$:



Weierstrass function:



Fractal interpolation function (the 2-ml spectrum is not convex here and the red line is its convex envelop):



The 2ml formalism

Theorem (Lévy Véhel/Seuret)

Let $f \in \mathcal{S}'(\mathbb{R})$. The 2-microlocal frontier of f at any x_0 is the Legendre transform of $-\chi$, *i.e.*:

$$\sigma(s') = (-\chi)^*(s') := \inf_{\alpha \in [0,1]} (\alpha s' + \chi(\alpha))$$

Remark: By inverse Legendre transform, one gets:

$$-\tilde{\chi}(\rho) = \inf_{s' \in \mathbb{R}} \{\rho s' - \sigma(s')\}.$$

Discrete-time 2ml spectrum and formalism

One can define a 2-ml spectrum using the discrete wavelet coefficients instead of the continuous ones. The procedure is the same, and one obtains again the 2-ml frontier as the Legendre transform of $-\chi$. In addition, one can show that the spectrum so obtained *does not depend on the orthonormal wavelet basis*.

Note that χ contains more information that the 2-ml frontier, as χ does not have to be convex. Moreover, this additional information is intrinsic to the analyzed function.

The 2ml spectrum and the regularity exponents

A major advantage of χ is that it allows to compute the exponents very easily:

Proposition

Assume $\alpha_l(x_0) > 0$. Then:

1. $\alpha_l(x_0) = \inf\{\chi_{x_0}(\rho) : \rho \in [0, 1]\}$.

2. $\alpha(x_0) = \inf\{\frac{\chi_{x_0}(\rho)}{\rho} : \rho \in (0, 1]\}$.

3. $\beta_w(x_0) = \chi_{x_0}(1) \in [0, +\infty]$.

4. If $\alpha(x_0) < +\infty$, then $\beta_o(x_0)$ is the smallest number β that satisfies

$$\chi_{x_0}\left(\frac{1}{\beta+1}\right) = \frac{\alpha(x_0)}{\beta+1}.$$

5. $\alpha_l(x_0) = \alpha(x_0) \Rightarrow \alpha(x_0) = \chi_{x_0}(1)$ and $\beta_o(x_0) = 0$.

6. $\alpha_l(x_0) = \alpha(x_0) \Rightarrow \chi_{x_0}(\rho) \geq \alpha(x_0)$, $\forall \rho$.

Item 2 shows how both the magnitude and the location of the large wavelet coefficients play a role in determining the pointwise Hölder exponent.

Item 4 allows to explain the "ringing" effect observed in wavelet hard-thresholding denoising.

Item 5 is crucial in the correct use of standard estimation procedures.

The 2ml formalism and the multifractal formalism

Recalls: Large deviation multifractal spectrum

For a function X, the large deviation spectrum is defined as:

$$f_g(\alpha) = \lim_{\varepsilon \to 0^+} f_g^{\varepsilon}(\alpha)$$

where:

$$\begin{aligned} f_g^{\varepsilon} &= \liminf_{n\to\infty} \frac{\log N_n^{\varepsilon}(\alpha)}{n} \\ N_n^{\varepsilon}(\alpha) &= \#\{k : \alpha - \varepsilon \le \alpha_n^k \le \alpha + \varepsilon\} \\ \alpha_n^k &= \frac{\log |Y_n^k|}{-n} \end{aligned}$$

and:

$$\begin{aligned} Y_n^k &= \text{increments of } X \text{ in } I_n^k = [k2^{-n}, (k+1)2^{-n}], \text{ or} \\ &= \text{oscillation of } X \text{ in } I_n^k, \text{ or} \\ &= \text{wavelet coefficient of } X \text{ at scale } n \text{ and location } k. \end{aligned}$$

Weak multifractal formalism

The (weak) multifractal formalism, when it holds, states that f_g is the Legendre transform of τ:

$$f_g(\alpha) = \tau^*(\alpha)$$

where

$$\begin{aligned} \tau(q) &= \liminf_{n\to\infty} \frac{\log S_n(q)}{-n} \\ S_n(q) &= \sum_{k=0}^{2^n-1} |Y_n^k|^q \end{aligned}$$

Under mild conditions (for instance supp f_g bounded) one always has:

$$\tau(q) = (f_g)^*(q)$$

2ml analysis = "multifractal analysis at one point"

X is said to be multifractal if:

- $f_g(\alpha) \geq 0$ for more than one exponent α: This means that different regularity behaviours occur along time,

and/or

- $\tau(q)$ is not a straight line: This means that it is not possible to predict the behaviour of $S_n(q)$ from the one of $S_n(q_0)$ (recall that, heuristically, $S_n(q) \sim 2^{-n\tau(q)}$).

Multisingularity

We say that X is **multisingular** at x if:

- $\chi_x(\alpha) < +\infty$ for more than one exponent: This means that different decays of wavelet coefficients occur on different curves $(b) = x_0 + a^{\alpha}$.

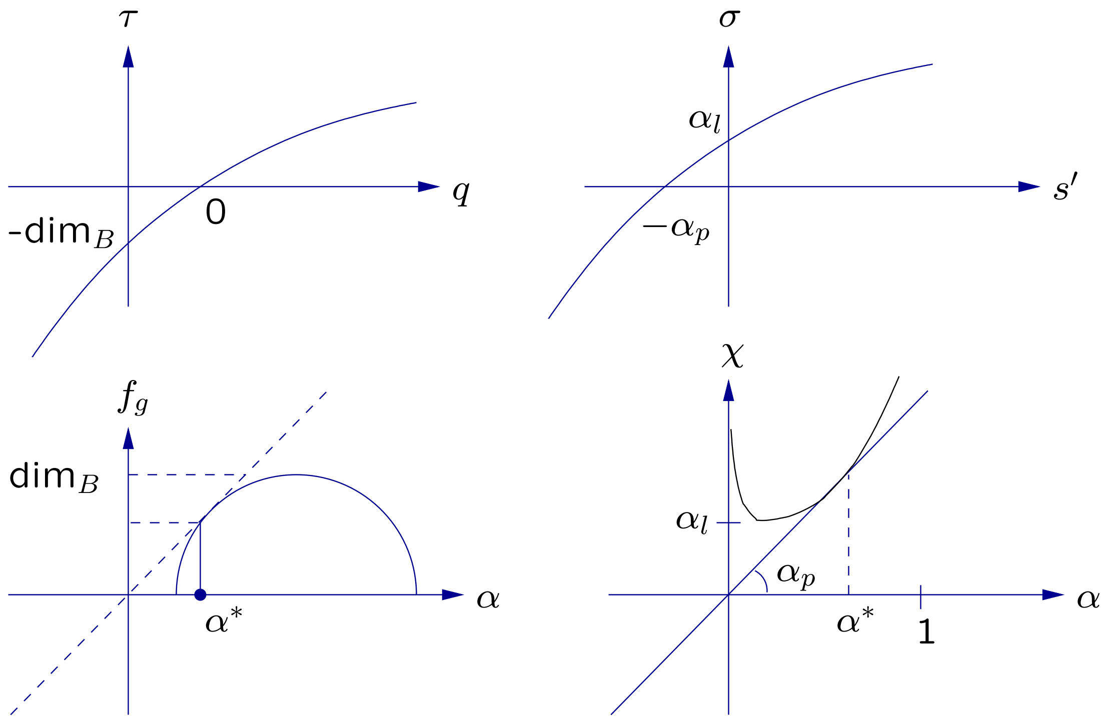
- $\sigma_x(s')$ is not a straight line: This means that one cannot predict the variation of α_p through integration of order ε, even if its variation through integration of order $\varepsilon' \neq \varepsilon$ is known.

Formal correspondences between 2-microlocal and multifractal analysis.

Multifractal Analysis	2-microlocal Analysis
q	s' or s
$\tau(q)$	$\sigma(s')$ or $s'(s)$
α	ρ
$f_g(\alpha)$	$\chi(\rho)$
$\dim_B(Supp(\mu))$	$-\alpha_l$
$D_q = \frac{\tau(q)}{q-1}$	$\Delta(s') = \frac{\sigma(s')}{s'+\alpha}$
Information Dimension $\alpha^* = D_1$	Information Exponent $\rho^* = \sigma'(-\alpha)$

Note that $\rho^* = \frac{1}{1+\beta_o}$. It has the following intuitive meaning: There always exists a single curve $|b - x_0| = a^{\rho^*}$, with $\rho^* = \sigma'(-\alpha)$, that dictates the rate of increase of α, which equals $\frac{1}{\rho^*}$.

Graphical comparison between τ and σ, and f_g and χ.



Further links between 2ml and multifractal analysis

Let $E_\alpha = \{x : \alpha(x) = \alpha\}$, and define the Hausdorff spectrum f_h of X as $f_h(\alpha) = d_H(E_\alpha)$, where d_H is the Hausdorff dimension.

Proposition

Assume $X \in C^\gamma$ for some $\gamma > 0$. Then:

- $\sup_{x\in\mathbb{R}} \chi_x(0) = +\infty \Rightarrow$ Supp(f_h) unbounded.

- $E_\infty = \{x : \alpha(x) = +\infty\}$ is included in $\{x : X$ is monosingular at $x\}$.

- supp(f_h) bounded $\Rightarrow X$ is everywhere multisingular.

Part 2: Stochastic 2-microlocal analysis. The fine regularity of Gaussian processes (joint work with Erick Herbin)

Motivations

- Stochastic processes are heavily used in applications as models for real phenomena : Internet traffic, financial records, medical and radar images, ... One would like to characterize the regularity of the models, and how it evolves under various processing.

- 2-ml analysis has been invented to track the singularities of solutions of PDEs. It seems natural to generalize it in the context of Stochastic Differential Equations.

Stochastic 2-microlocal analysis

For simplicity, we shall restrict to the case of continuous nowhere differentiable processes. Recall that in this case, $0 \leq s + s' < 1, s \geq 0, -1 \leq s' \leq 1$, and the function f belongs to the space $C_{x_0}^{s,s'}$ iff:

$$|f(x) - f(y)| \leq C|x - y|^{s+s'}(|x - y| + |x - x_0|)^{-s'}$$

for all (x, y) in a neighbourhood of x_0.

The stochastic exponents and the stochastic 2-ml frontier

Let $X = \left\{X_t;\ t \in \mathbf{R}_+^N\right\}$ be a continuous Gaussian process.

Definition

Let $t_0 \in \mathbf{R}_+^N$. The pointwise and local Hölder exponents of X at t_0 are defined as:

$$\boldsymbol{\alpha}(t_0) = \sup\left\{\alpha; \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{|X_t - X_s|}{\rho^\alpha} < \infty\right\}$$

$$\boldsymbol{\alpha}_l(t_0) = \sup\left\{\alpha; \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{|X_t - X_s|}{\|t - s\|^\alpha} < \infty\right\}$$

The 2-*ml frontier* of X at t_0 is the random function $s' \mapsto \boldsymbol{\sigma}_{t_0}(s')$, defined for $s' \in (-\infty; 0)$:

$$\boldsymbol{\sigma}_{t_0}(s') = \sup \left\{ \sigma;\ \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{|X_t - X_s|}{\|t-s\|^{\sigma} \rho^{-s'}} < \infty \right\}$$

For a given realization ω,

$$X_t(\omega) \in C_t^{s,s'} \Longleftrightarrow s + s' < \boldsymbol{\sigma}_{t_0}(s')$$

.

Deterministic versions of the local exponent, 2-microlocal spaces and 2-microlocal frontier

We seek to obtain results such as: $X \in C_{x_0}^{s,s'}$ almost surely. In that view, we set the following definitions.

Definition: Deterministic local exponent

$$\underline{\alpha_l}(t_0) = \sup \left\{ \alpha;\ \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{E\left[X_t - X_s\right]^2}{\|t-s\|^{2\alpha}} < \infty \right\}$$

Definition: Deterministic 2-microlocal space

A Gaussian process X is said to belong to $\mathbb{C}_{t_0}^{s,s'}$ for a fixed $t_0 \in \mathbf{R}_+^N$ and some s, s' such that

$$\left\{ \begin{array}{c} 0 < \sigma = s + s' < 1 \\ s < 1 \\ s' < 0 \end{array} \right.$$

if

$$\limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{E\left[X_t - X_s\right]^2}{\|t-s\|^{2\sigma} \rho^{-2s'}} < +\infty$$

Definition: Deterministic 2-microlocal frontier

$$\begin{aligned} \underline{\sigma}_{t_0}(s') &= \sup \left\{ \sigma;\ \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{E\left[X_t - X_s\right]^2}{\|t-s\|^{2\sigma} \rho^{-2s'}} < \infty \right\} \qquad (1) \\ &= \sup \left\{ \sigma;\ X \in \mathbb{C}_{t_0}^{\sigma - s', s'} \right\} \end{aligned}$$

The almost sure 2-microlocal frontier: Results at one point

Theorem (Herbin/Lévy Véhel)

Let $X = \left\{X_t;\ t \in \mathbf{R}_+^N\right\}$ be a continuous Gaussian process. For any $t_0 \in \mathbf{R}_+^N$, the 2-microlocal frontier of X at t_0 is almost surely equal to the function $s' \mapsto \underline{\sigma}_{t_0}(s')$.

Corollary

For any $t_0 \in \mathbf{R}_+^N$, the pointwise Hölder exponent of X at t_0 is almost surely equal to $-\inf\{s' : \underline{\sigma}_{t_0}(s') \geq 0\}$, provided $\underline{\sigma}_{t_0}(0) > 0$.

For any $t_0 \in \mathbf{R}_+^N$, the local Hölder exponent of X at t_0 is almost surely equal to $\underline{\sigma}_{t_0}(0)$, provided $\underline{\sigma}_{t_0}(0) > 0$.

Uniform almost sure results on $\mathbf{R}_+^N$

Theorem (Herbin/Lévy Véhel)

Let $X = \left\{X_t;\ t \in \mathbf{R}_+^N\right\}$ be a continuous Gaussian process. Assume that the function $t_0 \mapsto \liminf_{u\to t_0} \underline{\alpha_l}(u)$ is positive. Then, almost surely

$$\forall t_0 \in \mathbf{R}_+^N;\ \liminf_{u\to t_0} \underline{\alpha_l}(u) \leq \boldsymbol{\alpha}_l(t_0) \leq \limsup_{u\to t_0} \underline{\alpha_l}(u)$$

Corollary

Let $X = \left\{X_t;\ t \in \mathbf{R}_+^N\right\}$ be a continuous Gaussian process. Assume that the function $t_0 \mapsto \underline{\alpha_l}(t_0)$ is continuous and positive. Then, almost surely

$$\forall t_0 \in \mathbf{R}_+^N;\ \boldsymbol{\alpha}_l(t_0) = \underline{\alpha_l}(t_0)$$

Corollary

Let $X = \left\{X_t;\ t \in \mathbf{R}_+^N\right\}$ be a Gaussian process. Assume that the function $t_0 \mapsto \underline{\alpha_l}(t_0)$ is continuous and positive. Then, almost surely

$$\forall t_0 \in \mathbf{R}_+^N, \forall s' < 0;\ \underline{\alpha_l}(t_0) + s' \leq \boldsymbol{\sigma}_{t_0}(s') \leq \limsup_{u \to t_0} \underline{\sigma}_u(s')$$

These results yield only bounds for the uniform almost sure 2-microlocal frontier.

In specific cases, we are able to obtain a sharp result: This happens for instance for fractional Brownian motion and regular multifractional Brownian motion.

Application (I): Fractional Brownian motion (fBm)

Fractional Brownian motion was introduced by Kolmogorov and first studied in a fractal framework by Mandelbrot. Many results are known about its regularity (Adler, Kahane,. . .)

Definition: Fractional Brownian motion is the continuous zero-mean Gaussian process $B^H = \left\{B_t^H; t \in \mathbf{R}_+\right\}$ such that for all $s, t \in \mathbf{R}_+$,

$$E\left[B_t^H - B_s^H\right]^2 = |t-s|^{2H} \tag{2}$$

where $H \in (0, 1)$.

The 2-ml frontier of fBm

A direct application of the results above yield:

Proposition

Almost surely, the 2-microlocal frontier at any t_0 of the fractional Brownian motion in the region

$$\begin{cases} -1 < s' < 0 \\ 0 < \sigma < 1 + s' \end{cases}$$

is equal to the line $\sigma = H + s'$.

Corollary

The local and pointwise Hölder exponents satisfy almost surely

$$\forall t_0 \in \mathbf{R}_+;\ \boldsymbol{\alpha}_l(t_0) = \boldsymbol{\alpha}(t_0) = H$$

Application (II): Multifractional Brownian motion (mBm)

The local regularity of fBm is constant along the paths.

This is boring.

A natural extension is to substitute the constant parameter H with a function $t \mapsto H(t)$ taking values in $(0, 1)$. This leads to the definition of multifractional Brownian motion.

mBm: Moving Average Representation

Definition (Lévy Véhel/Peltier): Let $H : [0, \infty) \longrightarrow [a, b] \subset (0, 1)$ be a Hölder function of exponent $\beta > 0$. For $t \geq 0$, the following random function is called multifractional Brownian motion with functional parameter H:

$$W_{H(t)}(t) = \int_{-\infty}^{0} [(t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2}] dW(s)$$

$$+ \int_{0}^{t} (t-s)^{H(t)-1/2} dW(s),$$

When $H(t) = H$ for all t, mBm is just fBm of exponent H.

Covariance of mBm

Let $X(t)$ be a standard mBm (i.e. such that the variance at time 1 is 1) with functional parameter $H(t)$. Then:

$$E(X(t)X(s)) =$$

$$D(H(t),H(s))(t^{H(t)+H(s)} + s^{H(t)+H(s)} - |t-s|^{H(t)+H(s)})$$

where

$$D(x,y) = \frac{\sqrt{\Gamma(2x+1)\Gamma(2y+1)\sin(\pi x)\sin(\pi y)}}{2\Gamma(x+y+1)sin(\pi(x+y)/2)}$$

Main properties of mBm

From the definition, one sees that mBm is a zero mean Gaussian process whose increments are in general neither independent nor stationary.

(recall: fBm has stationary correlated increments for $H \neq 1/2$).

The increments of mBm exhibit long range dependance for all H non constant

(recall: The increments of fBm have long range dependance for $H > 1/2$).

Recalls: Long range dependance

Let $Y(t)$ be a stationary second-order process.

The correlation function $\rho(k) = E(Y(t)Y(t+k))$ only depends on the lag k. A possible definition of long range dependence is:

Definition

$Y(t)$ is said to have long range dependence if

$$\sum_{0}^{+\infty} |\rho(k)| = +\infty$$

Long range dependance for non-stationary processes

The increments of mBm are not stationary. The definition above maybe adapted as follows:

Definition

Let $Y(t)$ be a second-order process. $Y(t)$ is said to have long range dependence if

$$\forall \delta > 0, \forall s \geq 0, \sum_{0}^{+\infty} |cor_Y(s, s + k\delta)| = +\infty$$

Behaviour of the covariance of mBm for small increments

For all fixed t_0 and $\rho > 0$, there exist positive constants $K(t_0)$ and $L(t_0)$ such that, for all $s, t \in B(t_0, \rho)$:

$$E\left[X_t - X_s\right]^2 = K(t_0)\|t-s\|^{2H(t)} + L(t_0)\left[H(t) - H(s)\right]^2 \\ +o\left(\|t-s\|^{2H(t)}\right) + o\left(H(t) - H(s)\right)^2$$

This approximation, together with the fact that mBm is Hölder-continuous but not differentiable, allows to compute the almost sure 2-microlocal frontier of the mBm at any point t_0.

Pointwise almost sure 2-microlocal frontier of mBm

Proposition

The 2-microlocal frontier of the multifractional Brownian motion in the region

$$\begin{cases} -1 < s' < 0 \\ 0 < \sigma < 1 + s' \end{cases}$$

is, at any fixed t_0, almost surely equal to the "minimum" of the 2-ml frontier of H at t_0 and the line $\sigma = H(t_0) + s'$:

$$\boldsymbol{\sigma}_{t_0}(s') = (H(t_0) + s') \wedge \beta_{t_0}(s')$$

where $\beta_{t_0}(s')$ denotes the 2-microlocal frontier of the deterministic function H at t_0.

Pointwise almost sure Hölder exponents of mBm

Corollary

At any t_0, the pointwise and local Hölder exponents of the multifractional Brownian motion verify almost surely:

$$\boldsymbol{\alpha}(t_0) = H(t_0) \wedge \beta(t_0)$$

$$\boldsymbol{\alpha}_l(t_0) = H(t_0) \wedge \tilde{\beta}(t_0)$$

where $\beta(t_0)$ and $\tilde{\beta}(t_0)$ denote the pointwise and local Hölder exponents of H at t_0.

Uniform almost sure local Hölder exponent of mBm

Under some assumptions on the function H or its regularity, uniform results hold.

Proposition

Let $X = \left\{X_t;\ t \in \mathbf{R}_+\right\}$ be a multifractional Brownian motion such that the function $t \mapsto \tilde{\beta}(t)$, where $\tilde{\beta}(t)$ is the local Hölder exponent of H at t, is continuous on some open interval I. Then the local Hölder exponent of X satisfy almost surely

$$\forall t \in I:\ \boldsymbol{\alpha}_l(t) = H(t) \wedge \tilde{\beta}(t)$$

Uniform almost sure 2-ml frontier of the *regular* mBm

Theorem

Let $X = \left\{X_t;\ t \in \mathbf{R}_+\right\}$ be a multifractional Brownian motion such that the function H satisfy, for some open interval I,

$$\forall t \in I : H(t) < \tilde{\beta}(t)$$

where $\tilde{\beta}(t)$ is the local Hölder exponent of H at t.
Then, almost surely, the 2-microlocal frontier at any $t_0 \in I$ of X in the region

$$\begin{cases} -1 < s' < 0 \\ 0 < \sigma < 1 + s' \end{cases}$$

is equal to the line $\sigma = H(t_0) + s'$.

In particular, almost surely, for all $t_0 \in I$, $\boldsymbol{\alpha}(t_0) = \boldsymbol{\alpha}_l(t_0) = H(t_0)$.

Case where H verifies some global regularity condition

Theorem

Assume that, for all $t_0 \in I$,

$$\limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{|H(t) - H(s)|}{\|t - s\|^{\beta_{t_0}} \rho^{-s'}} < \infty$$

Then, almost surely, for all $t_0 \in I$,

$$\boldsymbol{\sigma}_{t_0}(s') = (H(t_0) + s') \wedge \beta_{t_0}(s')$$

and

$$\boldsymbol{\alpha}(t_0) = H(t_0) \wedge \beta(t_0)$$

$$\boldsymbol{\alpha}_l(t_0) = H(t_0) \wedge \tilde{\beta}(t_0)$$

where $\beta(t_0)$ and $\tilde{\beta}(t_0)$ denote the pointwise and local Hölder exponents of H at t_0.

Application (III): Linear Stochastic Differential Equations

Consider the simple linear stochastic differential equation (SDE):

$$dX_t = \mu_t X_t dB_t + b_t dt$$

where the deterministic functions μ and b satisfy the usual conditions ensuring existence and uniqueness of the solution. Then:

$$E\left[X_t - X_s\right]^2 = \lambda(t) - \lambda(s) - 2\psi(s)(\psi(t) - \psi(s))$$

with

$$\psi(t) = \int_0^t b(u)du$$

and

$$\lambda(t) = K(t) \exp\left(\int_0^t \mu^2(u)du\right)$$

$$K'(t) = \frac{2\psi(t)\psi'(t)}{\exp(\int_0^t \mu^2(u)du)}$$

Almost sure frontier of the solution

It is easy to deduce the 2-ml frontier of λ and ψ from the ones of μ and b. Using the formula describing the variance of $(X_t - X_s)$, one gets:

Proposition

Assume $\mu(t) \neq 0$ and $b(t) \neq 0$. Then, the 2-ml frontier at t of the solution X of the previous SDE verifies almost surely:

$$\boldsymbol{\sigma}_X(s') = \frac{1}{2}\left(\sigma_\mu(2s') \wedge \sigma_b(2s')\right) + \frac{1}{2}$$

References

Part I: *The 2-Microlocal Formalism*, available at: http://fractales.inria.fr, and references therein.

Part II: *Fine analysis of the regularity of Gaussian processes: Stochastic 2-microlocal analysis*, available at: http://fractales.inria.fr, and references therein.